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Rational expectations can preclude trades*

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Abstract. We reconsider the no trade theorem in an exchange economy where the traders have non-partition information. By introducing a new concept, *rationality of expectations*, we show some versions of the theorem different from previous works, such as Geanakoplos (http://cowles.econ.yale.edu, 1989). We also reexamine a standard assumption of the no trade theorem: *the common prior assumption*.

Key words: no trade theorem, ex ante Pareto optimum, common knowledge, rational expectations equilibrium

1. Introduction

The no trade theorem has shown that new information will not give the traders any incentive to trade when their initial endowments are allocated ex ante Pareto-optimally. In this theorem, there are two standard assumptions: (1) the

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partitional information structure, and (2) the common prior assumption. This paper explores the extent to which these two assumptions are generalized in the theorem.

In recent years, several investigators have already generalized the assumptions in this theorem. For (1), Geanakoplos [3] neatly analyzes non-partition information structure[1] with the introduction of a new concept, *positive balancedness*. With this concept, he examines several classes of non-partition information and the relations among them, and characterizes Nash equilibrium and rational expectations equilibrium in those classes.

Our paper discusses similar issues, but captures different features from his analysis with a new concept, *rationality of expectations*. This concept means that each trader knows his own expected utility. As shown later, this requirement does not necessarily imply either partitional information structure or positive balancedness. Moreover it does not require that traders are risk-neutral or risk-averse, which is usually assumed in this literature (c.f. [7,15]).

We do not need (2), the common prior assumption, although recent research shows that the common prior gives a necessary and sufficient condition for the no trader theorem (See [2,8,11,14]). Among those authors, Morris [8] explores different varieties of heterogeneous prior beliefs. We comment on heterogeneous priors in our model below.

Several variations of the no trade theorem have been developed. Neeman [10] applies it in the case of p-beliefs, Luo and Ma [5] in the non-expected utility case, Morris and Skiadas [9] in the case of rationalizable trades, and so on. Our model applies it to expected utility and rational expectations equilibrium, and therefore uses the standard setting of the original as Milgrom and Stokey [7] and Sebenius and Geanakoplos [15].

This paper is organized as follows: In Sect. 2 we define an economy with non-partition information structure and rational expectations equilibrium in our economy. The key notion, rationality of expectations, is defined in this section. In Sect. 3 we show two extended no trade theorems, and we comment on welfare of the rational expectations equilibrium in our economy. In Sect. 4, we give an example to compare with Geanakoplos [3]. In the example, we consider non-partition information different from that of Geanakoplos. Finally Sect. 5 gives comments on the common prior assumption.

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[1] Brandenburger et al. [1] analyze correlated equilibrium in games with non-partition information. In addition, Samet [13], Rubinstein and Wolinsky [12], Matsuhisa and Kamiyama [6], and others show the Aumann's disagreement theorem in the non-partition information.

2. Model of an exchange economy

Let Ω be a non-empty *finite* set called a *state space* and let 2^{Ω} denote the field of all subsets of Ω. Each member of 2^{Ω} is called an *event* and each element of Ω called a *state*. We consider the set N of n traders; i.e., $N = \{1, 2, \dots, n\}$.

2.1. Information and knowledge

We define i's possible correspondence $P_i : \Omega \to 2^{\Omega} \setminus \emptyset$ where $P_i(\omega)$ is interpreted as the set of all the states that trader i thinks are possible at ω. A special class of correspondences $(P_i)_{i \in N}$ is called *RT-information structure*[2] if the following two conditions are satisfied for every $i \in N$:

Ref : $\omega \in P_i(\omega)$ for every $\omega \in \Omega$.
Trn : $\xi \in P_i(\omega)$ implies $P_i(\xi) \subseteq P_i(\omega)$ for all $\xi, \omega \in \Omega$.

The possible correspondence gives rise to i's knowledge operator K_i defined by $K_i E = \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$, which is the event that i knows E. Then P_i satisfies **Ref** if and only if K_i satisfies ‘Truth’:

T : $K_i E \subseteq E$ for every $E \in 2^{\Omega}$.

It satisfies **Trn** if and only if K_i satisfies “positive introspection”:

4 : $K_i E \subseteq K_i K_i E$ for every $E \in 2^{\Omega}$.

The *common knowledge* operator K_C is defined by the infinite recursion of knowledge operators:

$$K_C E := \bigcap_{k=1,2,\dots} \bigcap_{\{i_1, i_2, \dots, i_k\} \subset N} K_{i_1} K_{i_2} \dots K_{i_k} E.$$

Given the RT-information structure $(P_i)_{i \in N}$, the *commonly* possible operator is the correspondence $M : \Omega \to 2^{\Omega}$ defined by

$$M(\omega) = \bigcup (P_{i_1}(P_{i_2}(\cdots P_{i_k}(\omega) \cdots))),$$

where the union ranges over all finite sequences of traders. We note that $\omega \in K_C E$ if and only if $M(\omega) \subseteq E$.[3]

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[2] The RT-information structure stands for the *reflexive* and *transitive* information structure. Geanakoplos [3] refers the former as *nondelusion* and the latter as *knowing that you know* (KTYK).

[3] See Samet [13] for details.

2.2. Economy with RT-information structure

We define *a pure exchange economy with RT-information structure* $\mathcal{E}$ as a tuple

$$\langle N, (\Omega, (P_i, \mu_i)_{i\in N}), (e_i, U_i)_{i\in N}\rangle,$$

which consists of the following structure and interpretations: There are l commodities at each state, and it is assumed that i's consumption set at each state is $\mathbb{R}^l_+$. Each trader i has a state-dependent endowment $e_i : \Omega \to \mathbb{R}^l_+$ with $\sum_{i\in N} e_i(\omega) > 0$ for all $\omega \in \Omega$, a quasi-concave von Neumann–Morgenstern utility function $U_i : \mathbb{R}^l_+ \times \Omega \to \mathbb{R}$, and a subjective prior μ_i on Ω with *full support*[4] for every $i \in N$. In our economy $\mathcal{E}$, we assume that i's utility function $U_i(\cdot, \omega)$ for each ω is continuous and strictly quasi-concave.

The traders trade according to a profile $\boldsymbol{t} = (t_i)_{i\in N}$ of functions t_i from Ω into $\mathbb{R}^l$. A trade is said to be *feasible* if, for all $i \in N$ and for all $\omega \in \Omega$, $e_i(\omega) + t_i(\omega) \geq 0$ and $\sum_{i\in N} t_i(\omega) \leq 0$. Given initial endowments $(e_i)_{i\in N}$ and any feasible trade $\boldsymbol{t} = (t_i)_{i\in N}$, we refer to $(e_i + t_i)_{i\in N}$ as an allocation $\boldsymbol{a} = (a_i)_{i\in N}$. Note that an allocation is $\sum_{i\in N} a_i(\omega) \leq \sum_{i\in N} e_i(\omega)$ for every $\omega \in \Omega$. We denote by $\mathcal{A}$ the set of all allocations and denote by $\mathcal{A}_i$ the projection of A onto player i's allocations.

For i's allocation $a_i \in \mathcal{A}_i$, each trader i has expectations; i's *ex ante* expectation is defined by $\mathbf{E}_i[U_i(a_i)] := \sum_{\omega\in\Omega} U_i(a_i(\omega), \omega)\mu_i(\omega)$. Then we define ex ante Pareto optimality as follows:

Definition 1. *The endowments $(e_i)_{i\in N}$ are said to be* ex ante Pareto-optimal *if there is no allocation $(a_i)_{i\in N}$ such that $\mathbf{E}_i[U_i(a_i)] \geq \mathbf{E}_i[U_i(e_i)]$ for every trader $i \in N$ with at least one strict inequality.*

For i's allocation $a_i \in \mathcal{A}_i$, we define i's interim expectation at $\omega \in \Omega$ as $\mathbf{E}_i[U_i(a_i)|P_i](\omega) := \sum_{\xi\in\Omega} U_i(a_i(\xi), \xi)\mu_i(\xi|P_i(\omega))$. Then we define the acceptability of i's trade as:

Definition 2. *Given a feasible trade $\boldsymbol{t} = (t_i)_{i\in N}$, t_i is* acceptable *for trader $i \in N$ at state $\omega \in \Omega$ if $\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega) \geq \mathbf{E}_i[U_i(e_i)|P_i](\omega)$.*

We denote by $Acp_i(t_i)$ the set of all the states in which t_i is acceptable for i, and denote $Acp(\boldsymbol{t}) := \bigcap_{i\in N} Acp_i(t_i)$. Furthermore we set the event of i's interim expectation for the trade t_i at ω:

$$[\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)] := \{\xi \in \Omega \mid \mathbf{E}_i[U_i(e_i + t_i)|P_i](\xi) = \mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)\}.$$

Given the event $[\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)]$, we denote $R_i(t_i) = \{\omega \in \Omega \mid P_i(\omega) \subseteq [\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)]\}$ and $R(\boldsymbol{t}) = \bigcap_{i\in N} R_i(t_i)$.

[4] I.e., $\mu_i(\omega) > 0$ for every $\omega \in \Omega$.

Definition 3. *A trader i is* rational about his expectation *for his trade t_i at ω if $\omega \in R_i(t_i)$; that is, $\omega \in K_i([\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega))$. A trader i is rational* everywhere *about his expectation for t_i if $R_i(t_i) = \Omega$.*

The event $R_i(t_i)$ means that trader i knows his expected gain from t_i at ω. Trader i is interpreted as knowing his interim expected utility at ω. If we consider the standard information structure of a partition on Ω, trader i is necessarily rational everywhere; i.e., $R_i(t_i) = \Omega$.

2.3. Price system and rational expectations equilibrium

A *price system* is a positive function $p : \Omega \to \mathbb{R}^l_{++}$. The *budget set* of a trader i at a state ω for a price system p is defined by $B_i(\omega, p) = \{a \in \mathbb{R}^l_+ \mid p(\omega) \cdot a \leqq p(\omega) \cdot e_i(\omega)\}$.

We denote $\Delta(p)(\omega) := \{\xi \in \Omega |\ p(\xi) = p(\omega)\}$ and $\Delta(p)$ the partition induced by p i.e., $\Delta(p) = \{\Delta(p)(\omega)|\ \omega \in \Omega\}$. When trader i learns from prices, his new information is represented by a mapping $\Delta(p) \cap P_i : \Omega \to 2^\Omega$ defined by $(\Delta(p) \cap P_i)(\omega) := \Delta(p)(\omega) \cap P_i(\omega)$. Note that $(\Delta(p) \cap P_i)_{i \in N}$, as well as $(P_i)_{i \in N}$, is *RT*-information structure.

Definition 4 (Geanakoplos [3]). *A* rational expectations equilibrium *for an economy $\mathcal{E}$ is a pair $(p, \boldsymbol{x})$, in which p is a price system and $\boldsymbol{x} = (x_i)_{i \in N}$ is an allocation satisfying the following conditions:*

RE 1 *For every $\omega \in \Omega$, $\sum_{i \in N} x_i(\omega) = \sum_{i \in N} e_i(\omega)$.*
RE 2 *For every $\omega \in \Omega$ and each $i \in N$, $x_i(\omega) \in B_i(\omega, p)$.*
RE 3 *If $P_i(\omega) = P_i(\xi)$ and $p(\omega) = p(\xi)$, then $x_i(\omega) = x_i(\xi)$ for trader $i \in N$ for any $\xi, \omega \in \Omega$.*
RE 4 *For each $i \in N$ and any mapping $y_i : \Omega \to \mathbb{R}^l_+$ with $y_i(\omega) \in B_i(\omega, p)$ for all $\omega \in \Omega$,*

$$\mathbf{E}_i[U_i(x_i)|\Delta(p) \cap P_i](\omega) \geqq \mathbf{E}_i[U_i(y_i)|\Delta(p) \cap P_i](\omega).$$

The profile $\boldsymbol{x} = (x_i)_{i \in N}$ is called a rational expectations equilibrium allocation.

For i's trade t_i, we set

$$R_i(p, t_i) := \{\omega \in \Omega |\ (\Delta(p) \cap P_i)(\omega) \subseteq [\mathbf{E}_i[U_i(e_i + t_i)|\Delta(p) \cap P_i](\omega)]\},$$

and denote $R(p, \boldsymbol{t}) = \bigcap_{i \in N} R_i(p, t_i)$. The set $R_i(p, t_i)$ is interpreted as the event that i knows his interim expectation for his trade t_i when he receives some new information from the price system p, and $R(p, \boldsymbol{t})$ is interpreted as the event that everyone knows his interim expectation for his trade with the price system p.

Definition 5. *A trader i is said to be* rational *about his expectation for t_i* with a price system *p at ω if $\omega \in R_i(p, t_i)$. All traders are* rational everywhere *about their expectations for $\boldsymbol{t}$ with p if $R(p, \boldsymbol{t}) = \Omega$.*

3. No trade theorems

In this section we shall give two extensions of the no trade theorem of Milgrom and Stokey [7]. In addition, we show the welfare of the rational expectations equilibrium.

3.1. No trade theorem with RT-information structure

The following is a direct extension of Milgrom and Stokey's theorem to an economy with RT-information structure, which will be proved in Appendix.

Theorem 1. *Let $\mathcal{E}$ be an economy with RT-information structure, and let $\boldsymbol{t} = (t_i)_{i \in N}$ be a feasible trade. Suppose that the initial endowments $(e_i)_{i \in N}$ are ex ante Pareto optimal. Then the traders can never agree to any non-null trade at each state where they commonly know both the acceptable trade $\boldsymbol{t} = (t_i)$ and where they are rational about their expectations for the trade; that is,*

$$\boldsymbol{t}(\omega) = \mathbf{0} \text{ at every } \omega \in K_C(Acp(\boldsymbol{t}) \cap R(\boldsymbol{t})).$$

To state this in a different way, we introduce the knowledge operator $K_i^{(p)}$ *associated with a price system p*, which is defined by $K_i^{(p)} E = \{\omega \in \Omega | (\Delta(p) \cap P_i)(\omega) \subseteq E\}$. The common knowledge operator $K_C^{(p)}$ associated with p is also defined by

$$K_C^{(p)} E := \bigcap_{k=1,2,\ldots} \bigcap_{\{i_1, i_2, \ldots, i_k\} \subset N} K_{i_1}^{(p)} K_{i_2}^{(p)} \ldots K_{i_k}^{(p)} E.$$

Then we obtain another no trade theorem with a price system p in the same way as Theorem 1.

Corollary 1. *Let $\mathcal{E}$ be an economy with RT-information structure. If $\boldsymbol{e} = (e_i)_{i \in N}$ is a rational expectations equilibrium allocation relative to some price system p with which all traders are rational everywhere about their expectations for the trade $\boldsymbol{t} = (t_i)_{i \in N}$, then the traders can never agree to any non-null trade at each state where they commonly know the acceptable feasible trade; that is,*

$$\boldsymbol{t}(\omega) = \mathbf{0} \text{ at every } \omega \in K_C^{(p)}(Acp(\boldsymbol{t})).$$

3.2. Welfare in an economy with knowledge

We examine the welfare of the rational expectations equilibrium in our economy. It is characterized from the viewpoint of ex ante optimality. This will be proved in Appendix as well as Theorem 1.

Proposition 1. *In an economy with RT-information structure $\mathcal{E}$, let an allocation $\boldsymbol{x} = (x_i)_{i \in N}$ be a rational expectations equilibrium allocation relative to some price system p with which all the traders are rational everywhere about their expectations with respect to $(x_i - e_i)_{i \in N}$. Then $\boldsymbol{x}$ is ex ante Pareto optimal.*

4. Example

We give an example to make clear the difference with Geanakoplos [3]. In our model, we impose reflexivity and transitivity on traders' information structure while Geanakoplos imposes reflexivity and positive balancedness as follows:

Definition 6. *The information structure* (Ω, P) *is called* positively balanced *with respect to* $E \subset \Omega$ *if there is a function* $\lambda : \underline{P} \to \mathbb{R}_+$ *such that*

$$\sum_{\substack{C \in \underline{P} \\ C \subset E}} \lambda(C) \chi_C(\omega) = \chi_E \ \textit{for all } \omega \in \Omega,$$

where $\underline{P} := \{F \in 2^{\Omega} | \ F = P(\omega) \text{ for some } \omega\}$*, and* χ_A *is the characteristic function of any set* $A \subset \Omega$*.*

Although positively balanced information structure is weaker than partitional structure, it does not necessarily imply RT-information structure.[5] Therefore our theorem under RT-information structure is obtained under a different setting in which the information structure is reflexive and transitive but not positively balanced. The following example illustrates a consequence of our theorem.

Example 1. Consider an economy $\mathcal{E}$ with RT-information structure where there is a single contingent commodity. The economy consists of: $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The endowments, information structure, traders' priors and utilities, and their trades are given as Table 1:

In this example, the RT-information structure is not positively balanced and the endowments are allocated ex ante Pareto-optimally. In addition, we do not specify the traders' attitudes toward risk like Geanakoplos [3], but unlike several other papers such as Milgrom and Stokey [7], or Sebenius and Geanakoplos [15]. This means that the crucial character of utility is strict quasi-concavity or monotonicity.

For the feasible trade $\boldsymbol{t} = (t_i)_{i \in N}$, $Acp(\boldsymbol{t}) = \Omega$ and then $K_C(Acp(\boldsymbol{t})) = \Omega$. Non-zero trades, however, occur at ω_1, ω_2, and ω_3. This is because $R(\boldsymbol{t}) = \{\omega_4\}$. That is, $K_C(Acp(\boldsymbol{t}) \cap R(\boldsymbol{t})) = \{\omega_4\}$. In this case, zero trade occurs at the state ω_4.

[5] See Geanakoplos [3, p.19] for these relations.

Table 1. Example 1

	Trader 1	Trader 2
(e_i)	$e_1(\omega) := \begin{cases} 5/2 & \text{for } \omega_1 \\ 1/3 & \text{for } \omega_2 \\ 1 & \text{for } \omega_3 \\ 2 & \text{for } \omega_4 \end{cases}$	$e_2(\omega) := \begin{cases} 1 & \text{for } \omega_1, \omega_2 \\ 5/2 & \text{for } \omega_3 \\ 1 & \text{for } \omega_4 \end{cases}$
(U_i)	$U_1(x,\omega) := \begin{cases} x & \text{for } \omega_1, \omega_2 \\ x^{\frac{4}{5}} & \text{for } \omega_3 \\ x^2 & \text{for } \omega_4 \end{cases}$	$U_2(x,\omega) := \begin{cases} x^{\frac{6}{5}} & \text{for } \omega_1 \\ x^2 & \text{for } \omega_2 \\ x & \text{for } \omega_3 \\ x^2 & \text{for } \omega_4 \end{cases}$
(P_i)	$P_1(\omega) := \begin{cases} \{\omega_1, \omega_3\} & \text{for } \omega_1 \\ \{\omega_2, \omega_3\} & \text{for } \omega_2 \\ \{\omega_3\} & \text{for } \omega_3 \\ \{\omega_4\} & \text{for } \omega_4 \end{cases}$	$P_2(\omega) := \begin{cases} \{\omega_1, \omega_2\} & \text{for } \omega_1 \\ \{\omega_2\} & \text{for } \omega_2 \\ \{\omega_2, \omega_3\} & \text{for } \omega_3 \\ \{\omega_4\} & \text{for } \omega_4 \end{cases}$
(μ_i)	$\mu_1(\omega) := \begin{cases} 1/2 & \text{for } \omega_1 \\ 1/3 & \text{for } \omega_2 \\ 1/12 & \text{for } \omega_3, \omega_4 \end{cases}$	$\mu_2(\omega) := \begin{cases} 1/6 & \text{for } \omega_1 \\ 1/2 & \text{for } \omega_2 \\ 1/6 & \text{for } \omega_3, \omega_4 \end{cases}$
(t_i)	$t_1(\omega) := \begin{cases} 3/5 & \text{for } \omega_1 \\ -2/15 & \text{for } \omega_2 \\ 4/5 & \text{for } \omega_3 \\ 0 & \text{for } \omega_4 \end{cases}$	$t_2(\omega) := \begin{cases} -3/5 & \text{for } \omega_1 \\ 2/15 & \text{for } \omega_2 \\ -4/5 & \text{for } \omega_3 \\ 0 & \text{for } \omega_4 \end{cases}$

On the whole, what role does the rationality of expectations play in our model? Since, under this concept, each trader knows his expected utility of a given trade, a relationship is stipulated between traders' information structure and expected gains. This approach is similar to the non-partition information technique of Aumann's disagreement theorem.

The technique is made clear by Rubinstein and Wolinsky [12] and Matsuhisa and Kamiyama [6], whose analyses are based on the decomposition of information structure of Samet [13]. However, their two analyses are slightly different from each other. Rubinstein and Wolinsky give a result relating two functions of 2^{Ω} between players, whereas Matsuhisa and Kamiyama analyze each player's function of 2^{Ω} with the same assumption as our rationality of expectations (Lemma 1 in Appendix). The latter approach enables us to analyze trader's interim expected utility from the ex ante viewpoint (Lemma 2 in our Appendix). Therefore we prove our no trade theorem with the rationality of expectations as an application of Samet's decomposition à la Matsuhisa and Kamiyama.

5. Concluding remarks

This paper has examined the no trade theorem under RT-information structure by introducing the concept of rationality of expectations. Although this situation has been investigated by Geanakoplos [3], our no trade theorem is shown under a slightly different setting as illustrated above, i.e., not positively balanced but RT-information structure. As stated in the Introduction, the common prior assumption is another standard assumption in the no trade theorem. Finally we comment on the relation between this assumption and our model.

Recent research shows that a common prior is a necessary and sufficient condition of the no trade result [2,8,11,14]. Among these authors, Morris shows the no trade result with heterogeneous priors in a general belief system ([8, p. 1336]).

In our framework, Morris's belief condition, called *the public consistent concordance*, means that, for any trader $i, j \in N$, $\mu_i(\xi|P_i(\omega)) = \mu_j(\xi|P_j(\omega))$ for any ξ, ω in a common knowledge event. Referencing to our example again, although $Acp(\boldsymbol{t})$ is a common knowledge event, any state except ω_4 is not public consistent concordant. Therefore, as shown by Morris [8, Corollary 3.2], there exists a common knowledge event that non-zero trade occurs from ex ante Pareto efficient endowments. Our result is consistent with Morris's under non-partition information structure.[6]

Appendix

Basic lemmas

In a decision set Ω, a function f of 2^{Ω} is said to be *preserved under difference* provided that, if $f(S) = f(T) = d$, then $f(T \setminus S) = d$ for all events S and T

[6] See Ng [11, Remark 2, p. 46].

with $S \subseteq T$. Furthermore the function f is said to satisfy the *sure thing principle* if $f(S \cup T) = d$ for two disjoint events S and T with $f(S) = f(T) = d$. When we consider the function $f_i(a_i) : 2^\Omega \to \mathbb{R}$ for $a_i \in \mathcal{A}_i$, which is defined by

$$f_i(a_i)(X) := \mathbf{E}_i[U_i(a_i)|X] = \sum_{\xi \in \Omega} U_i(a_i(\xi), \xi)\mu_i(\xi|X),$$

it is preserved under difference and satisfies the sure thing principle. Then we show the first lemma proved as the Fundamental lemma in Matsuhisa and Kamiyama [6].

Lemma 1. *Let P_i be i's RT-information structure and Π_i be the partition induced by P_i such that $\Pi_i(\omega) := \{\xi \in \Omega | \ P_i(\xi) = P_i(\omega)\}$. Then, if $P_i(\omega) \subseteq \{\xi \in \Omega | \ f(a_i)(P_i(\xi)) = f(a_i)(P_i(\omega))\}$ for $\omega \in \Omega$ and $a_i \in \mathcal{A}_i$, $f_i(a_i)(P_i(\omega)) = f_i(a_i)(\Pi_i(\xi))$ for every $\xi \in P_i(\omega)$.*

Let M be the common possible operator associated with K_C.

Lemma 2. *Let $\mathcal{E}$ be an economy with RT-information structure and $\boldsymbol{t} = (t_i)_{i \in N}$ be a feasible trade. If $\omega \in K_C(Acp_i(t_i) \cap R_i)$ for each $i \in N$ then the following equality is true:*

$$\mathbf{E}_i[U_i(t_i^* + e_i)|P_i](\omega) = \mathbf{E}_i[U_i(e_i)|P_i](\omega), \tag{1}$$

where the trade $\boldsymbol{t}^* = (t_i^*)_{i \in N}$ is defined by

$$t_i^*(\xi) := \begin{cases} t_i(\xi) & \text{if } \xi \in M(\omega), \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

Proof. We specify $\Pi_i(\omega) = \{\xi \in \Omega | \ P_i(\xi) = P_i(\omega)\}$ for every $\omega \in \Omega$. We can observe the two points: First $\boldsymbol{t}^* = (t_i^*)_{i \in N}$ is feasible because so is $\boldsymbol{t}$, and secondly $M(\omega) = \Pi_i(\xi_1) \cup \Pi_i(\xi_2) \cup \cdots \cup \Pi_i(\xi_K)$ for $\xi_k \in M(\omega)$ $(1 \leq k \leq K)$. We notice by Lemma 1 that, given $a_i \in \mathcal{A}_i$,

$$\mathbf{E}_i[U_i(a_i)| \ P_i)](\xi) = \mathbf{E}_i[U_i(a_i)| \ \Pi_i](\xi) \quad \text{for all } \xi \in M(\omega). \tag{3}$$

Then, it follows that

$$\begin{aligned} \mathbf{E}_i[U_i(t_i^* + e_i)] = & \sum_{k=1}^{K} \sum_{\xi \in \Pi_i(\xi_k)} U_i(t_i(\xi) + e_i(\xi), \xi)\mu_i(\xi) \\ & + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \mu_i(\Pi(\xi_k))\mathbf{E}_i[U_i(t_i + e_i)|P_i](\xi_k) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi) \\
&\geqq \sum_{k=1}^{K} \mu_i(\Pi_i(\xi_k))\mathbf{E}_i[U_i(e_i)|P_i](\xi_k) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi) \qquad (4) \\
&= \mathbf{E}_i[U_i(e_i)].
\end{aligned}
$$

Inequality (4) is owing to $\xi_k \in M(\omega) \subseteq Acp(t_i)$ for all k. That is, $P_i(\xi_k) \subseteq M(\omega) \subseteq Acp(t_i)$ for every $\xi_k \in M(\omega)$ $(1 \leq k \leq K)$.

Therefore, if equation (1) does not hold, inequality (4) holds strictly. This means that $\mathbf{E}_i[U_i(t_i^* + e_i)] \gneqq \mathbf{E}_i[U_i(e_i)]$, in contradiction to the assumption that $(e_i)_{i \in N}$ is ex ante Pareto optimal. □

Proof of Theorem 1

Suppose to the contrary that $t_i(\omega) \neq \mathbf{0}$ at some $\omega \in K_C(Acp(\boldsymbol{t}) \cap R(\boldsymbol{t}))$. We set $A_i := \{\omega \in K_C(Acp(\boldsymbol{t}) \cap R(\boldsymbol{t}))|\ t_i(\omega) \neq 0\}$. Then we define the trade $\boldsymbol{t}^* = (t_i)_{i \in N}$ in Lemma 2 as follows:

$$
t_i^*(\xi) := \begin{cases} \frac{t_i(\xi)}{2} & \text{if } \xi \in A_i, \\ 0 & \text{otherwise.} \end{cases} \qquad (5)
$$

Since $t_i(\xi)$ is feasible, so is t_i^*. Noting that $e_i + \frac{1}{2}t_i$ is a convex combination between e_i and $e_i + t_i$, it follows from $\omega \in A_i \subseteq K_C(Acp(t_i))$ and the quasi concavity of U_i that

$$
\mathbf{E}_i\left[U_i\left(e_i + \frac{1}{2}t_i\right)|P_i\right](\omega) \gneqq \mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega) \geq \mathbf{E}_i[U_i(e_i)|P_i](\omega),
$$

in contradiction to the ex ante Pareto optimality of $(e_i)_{i \in N}$ for the same reason as Lemma 2. □

Proof of Proposition 1

We set $\Pi_i(p)(\omega) := \{\xi \in \Omega|\ (\Delta(p) \cap P_i)(\xi) = (\Delta(p) \cap P_i)(\omega)\}$ for each $\omega \in \Omega$. Then $\Omega = \cup_{k=1}^{K} \Pi_i(p)(\omega_k)$. Since $\Delta(p) \cap P_i$ is i's information structure and $R_i(p, x_i) = \Omega$, it follows from Lemma 1 and **RE 4** that, for all $\xi \in \Pi_i(p)(\omega) \subseteq (\Delta(p) \cap P_i)(\omega)$,

$$\mathbf{E}_i[U_i(x_i)|(\Delta(p) \cap P_i)](\xi) = \mathbf{E}_i[U_i(x_i)|\Pi_i(p)](\xi)$$
$$\geq \mathbf{E}_i[U_i(e_i)|(\Delta(p) \cap P_i)](\xi) = \mathbf{E}_i[U_i(e_i)|\Pi_i(p)](\xi).$$

By adding up the above inequality over $\Pi_i(p)$, we obtain that, for all $i \in N$,

$$\begin{aligned} E_i[U_i(x_i)] &= \sum_{k=1}^{K} \mu_i(\Pi_i(p)(\omega_k))\mathbf{E}_i[U_i(x_i)|\Pi_i(p)](\omega_k) \\ &\geq \sum_{k=1}^{K} \mu_i(\Pi_i(p)(\omega_k))\mathbf{E}_i[U_i(e_i)|\Pi_i(p)](\omega_k) \\ &= E_i[U_i(e_i)]. \end{aligned}$$

□

References

1. Brandenburger, B., Dekel, E., Geanakoplos, J.: Correlated equilibrium with generalized information structures. Games Econ. Behav. **4**, 182–201 (1992)
2. Feinberg, Y.: Characterizing common priors in the form of posteriors. J. Econ. Theory **91**(2), 127–179 (2000)
3. Geanakoplos, J.: Game theory without partitions, and applications to speculation and consensus. Cowles Foundation Discussion Paper No. 914 (Available on http://cowles.econ.yale.edu) (1989)
4. Ishikawa, R.: Belief, Rationality, and Equilibrium in Game Theory, Ph. D. Dissertation, Hitotsubashi University (2003)
5. Luo, X., Ma, C.: Agreeing to disagree type results: a decision-theoretic approach. J. Math. Econ. **39**, 849–861 (2003)
6. Matsuhisa, T., Kamiyama, K.: Lattice structure of knowledge and agreeing to disagree. J. Math. Econ. **27**, 389–410 (1997)
7. Milgrom, P., Stokey, N.: Information, trade and common knowledge. J. Econ. Theory. **26**, 17–27 (1982)
8. Morris, S.: Trade with heterogeneous prior beliefs and asymmetric information. Econometrica **62**, 1327–1347 (1994)
9. Morris, S., Skiadas, C.: Rationalizable trade. Games Econ. Behav. **31**, 311–323 (2000)
10. Neeman, Z.: Common beliefs and the existence of speculative trade. Games Econ. Behav. **16**, 77–96 (1996)
11. Ng, M.-C.: On the duality between prior beliefs and trading demands. J. Econ. Theory **109**, 39–51 (2003)
12. Rubinstein, A., Wolinsky, A.: On the logic of "Agreeing to Disagree" type results. J. Econ. Theory **51**, 184–193 (1990)
13. Samet, D.: Ignoring ignorance and agreeing to disagree. J. Econ. Theory **52**, 190–207 (1990)
14. Samet, D.: Common priors and separation convex sets. Games Econ. Behav. **24**, 172–174 (1998)
15. Sebenius, J.K., Geanakoplos, J.: Don't bet on it: contingent agreements with asymmetric information. J. Am. Stat. Assoc. **78**(382), 424–426 (1983)